

The Hahn-Banach Theorem for Real Vector Spaces

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Abstract

The Hahn-Banach Theorem is one of the most fundamental results in functional analysis. We present a fully formal proof of two versions of the theorem, one for general linear spaces and another for normed spaces. This development is based on simply-typed classical set-theory, as provided by Isabelle/HOL.

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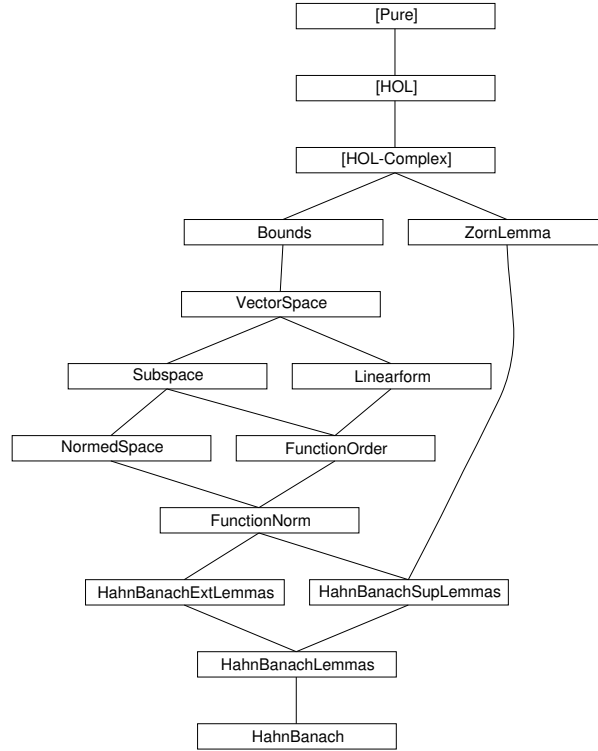
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1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser's textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.



Part I

Basic Notions

2 Bounds

```

theory Bounds imports Main Real begin

locale lub =
  fixes A and x
  assumes least [intro?]:  $(\bigwedge a. a \in A \implies a \leq b) \implies x \leq b$ 
  and upper [intro?]:  $a \in A \implies a \leq x$ 

lemmas [elim?] = lub.least lub.upper

constdefs
  the-lub :: 'a::order set  $\Rightarrow$  'a
  the-lub A  $\equiv$  The (lub A)

syntax (xsymbols)
  the-lub :: 'a::order set  $\Rightarrow$  'a    ( $\sqcup$ -[90] 90)

lemma the-lub-equality [elim?]:
  includes lub
  shows  $\sqcup A = (x::'a::order)$ 
   $\langle proof \rangle$ 

lemma the-lubI-ex:
  assumes ex:  $\exists x. \text{lub } A \ x$ 
  shows  $\text{lub } A (\sqcup A)$ 
   $\langle proof \rangle$ 

lemma lub-compat:  $\text{lub } A \ x = \text{isLub } UNIV \ A \ x$ 
   $\langle proof \rangle$ 

lemma real-complete:
  fixes A :: real set
  assumes nonempty:  $\exists a. a \in A$ 
  and ex-upper:  $\exists y. \forall a \in A. a \leq y$ 
  shows  $\exists x. \text{lub } A \ x$ 
   $\langle proof \rangle$ 

end

```

3 Vector spaces

```

theory VectorSpace imports Real Bounds Zorn begin

```

3.1 Signature

For the definition of real vector spaces a type $'a$ of the sort $\{plus, minus, zero\}$ is considered, on which a real scalar multiplication \cdot is declared.

consts

$prod :: real \Rightarrow 'a::\{plus, minus, zero\} \Rightarrow 'a$ (**infixr** $'(*)$ 70)

syntax (*xsymbols*)

$prod :: real \Rightarrow 'a \Rightarrow 'a$ (**infixr** \cdot 70)

syntax (*HTML output*)

$prod :: real \Rightarrow 'a \Rightarrow 'a$ (**infixr** \cdot 70)

3.2 Vector space laws

A *vector space* is a non-empty set V of elements from $'a$ with the following vector space laws: The set V is closed under addition and scalar multiplication, addition is associative and commutative; $-x$ is the inverse of x w. r. t. addition and 0 is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number 1 is the neutral element of scalar multiplication.

locale *vectorspace* = *var* $V +$

assumes *non-empty* [*iff*, *intro?*]: $V \neq \{\}$

and *add-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x + y \in V$

and *mult-closed* [*iff*]: $x \in V \Longrightarrow a \cdot x \in V$

and *add-assoc*: $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y) + z = x + (y + z)$

and *add-commute*: $x \in V \Longrightarrow y \in V \Longrightarrow x + y = y + x$

and *diff-self* [*simp*]: $x \in V \Longrightarrow x - x = 0$

and *add-zero-left* [*simp*]: $x \in V \Longrightarrow 0 + x = x$

and *add-mult-distrib1*: $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y$

and *add-mult-distrib2*: $x \in V \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x$

and *mult-assoc*: $x \in V \Longrightarrow (a * b) \cdot x = a \cdot (b \cdot x)$

and *mult-1* [*simp*]: $x \in V \Longrightarrow 1 \cdot x = x$

and *negate-eq1*: $x \in V \Longrightarrow -x = (-1) \cdot x$

and *diff-eq1*: $x \in V \Longrightarrow y \in V \Longrightarrow x - y = x + -y$

lemma (**in** *vectorspace*) *negate-eq2*: $x \in V \Longrightarrow (-1) \cdot x = -x$

<proof>

lemma (**in** *vectorspace*) *negate-eq2a*: $x \in V \Longrightarrow -1 \cdot x = -x$

<proof>

lemma (**in** *vectorspace*) *diff-eq2*: $x \in V \Longrightarrow y \in V \Longrightarrow x + -y = x - y$

<proof>

lemma (**in** *vectorspace*) *diff-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x - y \in V$

<proof>

lemma (**in** *vectorspace*) *neg-closed* [*iff*]: $x \in V \Longrightarrow -x \in V$

<proof>

lemma (**in** *vectorspace*) *add-left-commute*:

$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow x + (y + z) = y + (x + z)$

<proof>

theorems (in *vectorspace*) *add-ac* =
add-assoc add-commute add-left-commute

The existence of the zero element of a vector space follows from the non-emptiness of carrier set.

lemma (in *vectorspace*) *zero [iff]*: $0 \in V$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *add-zero-right [simp]*:
 $x \in V \implies x + 0 = x$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *mult-assoc2*:
 $x \in V \implies a \cdot b \cdot x = (a * b) \cdot x$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *diff-mult-distrib1*:
 $x \in V \implies y \in V \implies a \cdot (x - y) = a \cdot x - a \cdot y$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *diff-mult-distrib2*:
 $x \in V \implies (a - b) \cdot x = a \cdot x - (b \cdot x)$
 $\langle \text{proof} \rangle$

lemmas (in *vectorspace*) *distrib* =
add-mult-distrib1 add-mult-distrib2
diff-mult-distrib1 diff-mult-distrib2

Further derived laws:

lemma (in *vectorspace*) *mult-zero-left [simp]*:
 $x \in V \implies 0 \cdot x = 0$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *mult-zero-right [simp]*:
 $a \cdot 0 = (0 :: 'a)$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *minus-mult-cancel [simp]*:
 $x \in V \implies (- a) \cdot - x = a \cdot x$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *add-minus-left-eq-diff*:
 $x \in V \implies y \in V \implies - x + y = y - x$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *add-minus [simp]*:
 $x \in V \implies x + - x = 0$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *add-minus-left [simp]*:
 $x \in V \implies - x + x = 0$
 $\langle \text{proof} \rangle$

lemma (in *vectorspace*) *minus-minus* [simp]:

$$x \in V \implies -(-x) = x$$

<proof>

lemma (in *vectorspace*) *minus-zero* [simp]:

$$-(0::'a) = 0$$

<proof>

lemma (in *vectorspace*) *minus-zero-iff* [simp]:

$$x \in V \implies (-x = 0) = (x = 0)$$

<proof>

lemma (in *vectorspace*) *add-minus-cancel* [simp]:

$$x \in V \implies y \in V \implies x + (-x + y) = y$$

<proof>

lemma (in *vectorspace*) *minus-add-cancel* [simp]:

$$x \in V \implies y \in V \implies -x + (x + y) = y$$

<proof>

lemma (in *vectorspace*) *minus-add-distrib* [simp]:

$$x \in V \implies y \in V \implies -(x + y) = -x + -y$$

<proof>

lemma (in *vectorspace*) *diff-zero* [simp]:

$$x \in V \implies x - 0 = x$$

<proof>

lemma (in *vectorspace*) *diff-zero-right* [simp]:

$$x \in V \implies 0 - x = -x$$

<proof>

lemma (in *vectorspace*) *add-left-cancel*:

$$x \in V \implies y \in V \implies z \in V \implies (x + y = x + z) = (y = z)$$

<proof>

lemma (in *vectorspace*) *add-right-cancel*:

$$x \in V \implies y \in V \implies z \in V \implies (y + x = z + x) = (y = z)$$

<proof>

lemma (in *vectorspace*) *add-assoc-cong*:

$$\begin{aligned} x \in V \implies y \in V \implies x' \in V \implies y' \in V \implies z \in V \\ \implies x + y = x' + y' \implies x + (y + z) = x' + (y' + z) \end{aligned}$$

<proof>

lemma (in *vectorspace*) *mult-left-commute*:

$$x \in V \implies a \cdot b \cdot x = b \cdot a \cdot x$$

<proof>

lemma (in *vectorspace*) *mult-zero-uniq*:

$$x \in V \implies x \neq 0 \implies a \cdot x = 0 \implies a = 0$$

<proof>

lemma (in *vectorspace*) *mult-left-cancel*:

$x \in V \implies y \in V \implies a \neq 0 \implies (a \cdot x = a \cdot y) = (x = y)$
 <proof>

lemma (in *vectorspace*) *mult-right-cancel*:

$x \in V \implies x \neq 0 \implies (a \cdot x = b \cdot x) = (a = b)$
 <proof>

lemma (in *vectorspace*) *eq-diff-eq*:

$x \in V \implies y \in V \implies z \in V \implies (x = z - y) = (x + y = z)$
 <proof>

lemma (in *vectorspace*) *add-minus-eq-minus*:

$x \in V \implies y \in V \implies x + y = 0 \implies x = -y$
 <proof>

lemma (in *vectorspace*) *add-minus-eq*:

$x \in V \implies y \in V \implies x - y = 0 \implies x = y$
 <proof>

lemma (in *vectorspace*) *add-diff-swap*:

$a \in V \implies b \in V \implies c \in V \implies d \in V \implies a + b = c + d$
 $\implies a - c = d - b$
 <proof>

lemma (in *vectorspace*) *vs-add-cancel-21*:

$x \in V \implies y \in V \implies z \in V \implies u \in V$
 $\implies (x + (y + z) = y + u) = (x + z = u)$
 <proof>

lemma (in *vectorspace*) *add-cancel-end*:

$x \in V \implies y \in V \implies z \in V \implies (x + (y + z) = y) = (x = -z)$
 <proof>

end

4 Subspaces

theory *Subspace* **imports** *VectorSpace* **begin**

4.1 Definition

A non-empty subset U of a vector space V is a *subspace* of V , iff U is closed under addition and scalar multiplication.

locale *subspace* = var U + var V +

assumes *non-empty* [*iff*, *intro*]: $U \neq \{\}$

and *subset* [*iff*]: $U \subseteq V$

and *add-closed* [*iff*]: $x \in U \implies y \in U \implies x + y \in U$

and *mult-closed* [*iff*]: $x \in U \implies a \cdot x \in U$

declare *vectorspace.intro* [*intro?*] *subspace.intro* [*intro?*]

syntax (*symbols*)

$\text{subspace} :: 'a \text{ set} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$ (**infix** \trianglelefteq 50)

lemma *subspace-subset* [*elim*]: $U \trianglelefteq V \Longrightarrow U \subseteq V$
 $\langle \text{proof} \rangle$

lemma (**in** *subspace*) *subsetD* [*iff*]: $x \in U \Longrightarrow x \in V$
 $\langle \text{proof} \rangle$

lemma *subspaceD* [*elim*]: $U \trianglelefteq V \Longrightarrow x \in U \Longrightarrow x \in V$
 $\langle \text{proof} \rangle$

lemma *rev-subspaceD* [*elim?*]: $x \in U \Longrightarrow U \trianglelefteq V \Longrightarrow x \in V$
 $\langle \text{proof} \rangle$

lemma (**in** *subspace*) *diff-closed* [*iff*]:
includes *vectorspace*
shows $x \in U \Longrightarrow y \in U \Longrightarrow x - y \in U$
 $\langle \text{proof} \rangle$

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

lemma (**in** *subspace*) *zero* [*intro*]:
includes *vectorspace*
shows $0 \in U$
 $\langle \text{proof} \rangle$

lemma (**in** *subspace*) *neg-closed* [*iff*]:
includes *vectorspace*
shows $x \in U \Longrightarrow -x \in U$
 $\langle \text{proof} \rangle$

Further derived laws: every subspace is a vector space.

lemma (**in** *subspace*) *vectorspace* [*iff*]:
includes *vectorspace*
shows *vectorspace* U
 $\langle \text{proof} \rangle$

The subspace relation is reflexive.

lemma (**in** *vectorspace*) *subspace-refl* [*intro*]: $V \trianglelefteq V$
 $\langle \text{proof} \rangle$

The subspace relation is transitive.

lemma (**in** *vectorspace*) *subspace-trans* [*trans*]:
 $U \trianglelefteq V \Longrightarrow V \trianglelefteq W \Longrightarrow U \trianglelefteq W$
 $\langle \text{proof} \rangle$

4.2 Linear closure

The *linear closure* of a vector x is the set of all scalar multiples of x .

constdefs

$lin :: ('a::\{minus, plus, zero\}) \Rightarrow 'a\ set$
 $lin\ x \equiv \{a \cdot x \mid a.\ True\}$

lemma *linI* [*intro*]: $y = a \cdot x \Longrightarrow y \in lin\ x$
 $\langle proof \rangle$

lemma *linI'* [*iff*]: $a \cdot x \in lin\ x$
 $\langle proof \rangle$

lemma *linE* [*elim*]:
 $x \in lin\ v \Longrightarrow (\bigwedge a::real. x = a \cdot v \Longrightarrow C) \Longrightarrow C$
 $\langle proof \rangle$

Every vector is contained in its linear closure.

lemma (*in vectorspace*) *x-lin-x* [*iff*]: $x \in V \Longrightarrow x \in lin\ x$
 $\langle proof \rangle$

lemma (*in vectorspace*) *0-lin-x* [*iff*]: $x \in V \Longrightarrow 0 \in lin\ x$
 $\langle proof \rangle$

Any linear closure is a subspace.

lemma (*in vectorspace*) *lin-subspace* [*intro*]:
 $x \in V \Longrightarrow lin\ x \trianglelefteq V$
 $\langle proof \rangle$

Any linear closure is a vector space.

lemma (*in vectorspace*) *lin-vectorspace* [*intro*]:
 $x \in V \Longrightarrow vectorspace\ (lin\ x)$
 $\langle proof \rangle$

4.3 Sum of two vectorspaces

The *sum* of two vectorspaces U and V is the set of all sums of elements from U and V .

instance *set* :: (*plus*) *plus* $\langle proof \rangle$

defs (*overloaded*)
 $sum-def: U + V \equiv \{u + v \mid u.\ v.\ u \in U \wedge v \in V\}$

lemma *sumE* [*elim*]:
 $x \in U + V \Longrightarrow (\bigwedge u.\ v.\ x = u + v \Longrightarrow u \in U \Longrightarrow v \in V \Longrightarrow C) \Longrightarrow C$
 $\langle proof \rangle$

lemma *sumI* [*intro*]:
 $u \in U \Longrightarrow v \in V \Longrightarrow x = u + v \Longrightarrow x \in U + V$
 $\langle proof \rangle$

lemma *sumI'* [*intro*]:
 $u \in U \Longrightarrow v \in V \Longrightarrow u + v \in U + V$
 $\langle proof \rangle$

U is a subspace of $U + V$.

lemma *subspace-sum1* [iff]:
includes *vectorspace* $U + \text{vectorspace } V$
shows $U \leq U + V$
 <proof>

The sum of two subspaces is again a subspace.

lemma *sum-subspace* [intro?]:
includes *subspace* $U E + \text{vectorspace } E + \text{subspace } V E$
shows $U + V \leq E$
 <proof>

The sum of two subspaces is a vectorspace.

lemma *sum-vs* [intro?]:
 $U \leq E \implies V \leq E \implies \text{vectorspace } E \implies \text{vectorspace } (U + V)$
 <proof>

4.4 Direct sums

The sum of U and V is called *direct*, iff the zero element is the only common element of U and V . For every element x of the direct sum of U and V the decomposition in $x = u + v$ with $u \in U$ and $v \in V$ is unique.

lemma *decomp*:
includes *vectorspace* $E + \text{subspace } U E + \text{subspace } V E$
assumes *direct*: $U \cap V = \{0\}$
and $u1: u1 \in U$ **and** $u2: u2 \in U$
and $v1: v1 \in V$ **and** $v2: v2 \in V$
and *sum*: $u1 + v1 = u2 + v2$
shows $u1 = u2 \wedge v1 = v2$
 <proof>

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page ??): for any element $y + a \cdot x_0$ of the direct sum of a vectorspace H and the linear closure of x_0 the components $y \in H$ and a are uniquely determined.

lemma *decomp-H'*:
includes *vectorspace* $E + \text{subspace } H E$
assumes $y1: y1 \in H$ **and** $y2: y2 \in H$
and $x': x' \notin H \ x' \in E \ x' \neq 0$
and *eq*: $y1 + a1 \cdot x' = y2 + a2 \cdot x'$
shows $y1 = y2 \wedge a1 = a2$
 <proof>

Since for any element $y + a \cdot x'$ of the direct sum of a vectorspace H and the linear closure of x' the components $y \in H$ and a are unique, it follows from $y \in H$ that $a = 0$.

lemma *decomp-H'-H*:
includes *vectorspace* $E + \text{subspace } H E$
assumes $t: t \in H$
and $x': x' \notin H \ x' \in E \ x' \neq 0$
shows $(\text{SOME } (y, a). t = y + a \cdot x' \wedge y \in H) = (t, 0)$
 <proof>

The components $y \in H$ and a in $y + a \cdot x'$ are unique, so the function h' defined by $h'(y + a \cdot x') = h y + a \cdot \xi$ is definite.

```

lemma h'-definite:
  includes var  $H$ 
  assumes h'-def:
     $h' \equiv (\lambda x. \text{let } (y, a) = \text{SOME } (y, a). (x = y + a \cdot x' \wedge y \in H)$ 
       $\text{in } (h y) + a * xi)$ 
    and  $x: x = y + a \cdot x'$ 
  includes vectorspace  $E + \text{subspace } H E$ 
  assumes  $y: y \in H$ 
    and  $x': x' \notin H \ x' \in E \ x' \neq 0$ 
  shows  $h' x = h y + a * xi$ 
<proof>

end

```

5 Normed vector spaces

theory *NormedSpace* **imports** *Subspace* **begin**

5.1 Quasinorms

A *seminorm* $\|\cdot\|$ is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogenous and subadditive.

```

locale norm-syntax =
  fixes  $norm :: 'a \Rightarrow real$  ( $\|\cdot\|$ )

locale seminorm = var  $V + \text{norm-syntax} +$ 
  assumes ge-zero [iff?]:  $x \in V \Longrightarrow 0 \leq \|x\|$ 
    and abs-homogenous [iff?]:  $x \in V \Longrightarrow \|a \cdot x\| = |a| * \|x\|$ 
    and subadditive [iff?]:  $x \in V \Longrightarrow y \in V \Longrightarrow \|x + y\| \leq \|x\| + \|y\|$ 

declare seminorm.intro [intro?]

lemma (in seminorm) diff-subadditive:
  includes vectorspace
  shows  $x \in V \Longrightarrow y \in V \Longrightarrow \|x - y\| \leq \|x\| + \|y\|$ 
<proof>

lemma (in seminorm) minus:
  includes vectorspace
  shows  $x \in V \Longrightarrow \|- x\| = \|x\|$ 
<proof>

```

5.2 Norms

A *norm* $\|\cdot\|$ is a seminorm that maps only the 0 vector to 0 .

```

locale norm = seminorm +
  assumes zero-iff [iff]:  $x \in V \Longrightarrow (\|x\| = 0) = (x = 0)$ 

```

5.3 Normed vector spaces

A vector space together with a norm is called a *normed space*.

locale *normed-vectorspace* = *vectorspace* + *norm*

declare *normed-vectorspace.intro* [*intro?*]

lemma (**in** *normed-vectorspace*) *gt-zero* [*intro?*]:

$x \in V \implies x \neq 0 \implies 0 < \|x\|$
 $\langle \text{proof} \rangle$

Any subspace of a normed vector space is again a normed vectorspace.

lemma *subspace-normed-vs* [*intro?*]:

includes *subspace* *F E* + *normed-vectorspace E*

shows *normed-vectorspace F norm*

$\langle \text{proof} \rangle$

end

6 Linearforms

theory *Linearform* **imports** *VectorSpace* **begin**

A *linear form* is a function on a vector space into the reals that is additive and multiplicative.

locale *linearform* = *var V* + *var f* +

assumes *add* [*iff*]: $x \in V \implies y \in V \implies f (x + y) = f x + f y$

and *mult* [*iff*]: $x \in V \implies f (a \cdot x) = a * f x$

declare *linearform.intro* [*intro?*]

lemma (**in** *linearform*) *neg* [*iff*]:

includes *vectorspace*

shows $x \in V \implies f (- x) = - f x$

$\langle \text{proof} \rangle$

lemma (**in** *linearform*) *diff* [*iff*]:

includes *vectorspace*

shows $x \in V \implies y \in V \implies f (x - y) = f x - f y$

$\langle \text{proof} \rangle$

Every linear form yields 0 for the 0 vector.

lemma (**in** *linearform*) *zero* [*iff*]:

includes *vectorspace*

shows $f 0 = 0$

$\langle \text{proof} \rangle$

end

7 An order on functions

theory *FunctionOrder* **imports** *Subspace Linearform* **begin**

7.1 The graph of a function

We define the *graph* of a (real) function f with domain F as the set

$$\{(x, f x). x \in F\}$$

So we are modeling partial functions by specifying the domain and the mapping function. We use the term “function” also for its graph.

types $'a \text{ graph} = ('a \times \text{real}) \text{ set}$

constdefs

$\text{graph} :: 'a \text{ set} \Rightarrow ('a \Rightarrow \text{real}) \Rightarrow 'a \text{ graph}$
 $\text{graph } F f \equiv \{(x, f x) \mid x. x \in F\}$

lemma *graphI* [*intro*]: $x \in F \Longrightarrow (x, f x) \in \text{graph } F f$
 $\langle \text{proof} \rangle$

lemma *graphI2* [*intro?*]: $x \in F \Longrightarrow \exists t \in \text{graph } F f. t = (x, f x)$
 $\langle \text{proof} \rangle$

lemma *graphE* [*elim?*]:

$(x, y) \in \text{graph } F f \Longrightarrow (x \in F \Longrightarrow y = f x \Longrightarrow C) \Longrightarrow C$
 $\langle \text{proof} \rangle$

7.2 Functions ordered by domain extension

A function h' is an extension of h , iff the graph of h is a subset of the graph of h' .

lemma *graph-extI*:

$(\bigwedge x. x \in H \Longrightarrow h x = h' x) \Longrightarrow H \subseteq H'$
 $\Longrightarrow \text{graph } H h \subseteq \text{graph } H' h'$
 $\langle \text{proof} \rangle$

lemma *graph-extD1* [*dest?*]:

$\text{graph } H h \subseteq \text{graph } H' h' \Longrightarrow x \in H \Longrightarrow h x = h' x$
 $\langle \text{proof} \rangle$

lemma *graph-extD2* [*dest?*]:

$\text{graph } H h \subseteq \text{graph } H' h' \Longrightarrow H \subseteq H'$
 $\langle \text{proof} \rangle$

7.3 Domain and function of a graph

The inverse functions to *graph* are *domain* and *funct*.

constdefs

$\text{domain} :: 'a \text{ graph} \Rightarrow 'a \text{ set}$
 $\text{domain } g \equiv \{x. \exists y. (x, y) \in g\}$

$\text{funct} :: 'a \text{ graph} \Rightarrow ('a \Rightarrow \text{real})$
 $\text{funct } g \equiv \lambda x. (\text{SOME } y. (x, y) \in g)$

The following lemma states that g is the graph of a function if the relation induced by g is unique.

lemma *graph-domain-funct*:

assumes *uniq*: $\bigwedge x y z. (x, y) \in g \implies (x, z) \in g \implies z = y$

shows *graph* (domain g) (funct g) = g

<proof>

7.4 Norm-preserving extensions of a function

Given a linear form f on the space F and a seminorm p on E . The set of all linear extensions of f , to superspaces H of F , which are bounded by p , is defined as follows.

constdefs

norm-pres-extensions ::

$'a::\{\text{plus}, \text{minus}, \text{zero}\} \text{ set} \Rightarrow ('a \Rightarrow \text{real}) \Rightarrow 'a \text{ set} \Rightarrow ('a \Rightarrow \text{real})$
 $\Rightarrow 'a \text{ graph set}$

norm-pres-extensions $E p F f$

$\equiv \{g. \exists H h. g = \text{graph } H h$
 $\wedge \text{linearform } H h$
 $\wedge H \trianglelefteq E$
 $\wedge F \trianglelefteq H$
 $\wedge \text{graph } F f \subseteq \text{graph } H h$
 $\wedge (\forall x \in H. h x \leq p x)\}$

lemma *norm-pres-extensionE* [elim]:

$g \in \text{norm-pres-extensions } E p F f$
 $\implies (\bigwedge H h. g = \text{graph } H h \implies \text{linearform } H h$
 $\implies H \trianglelefteq E \implies F \trianglelefteq H \implies \text{graph } F f \subseteq \text{graph } H h$
 $\implies \forall x \in H. h x \leq p x \implies C) \implies C$
<proof>

lemma *norm-pres-extensionI2* [intro]:

$\text{linearform } H h \implies H \trianglelefteq E \implies F \trianglelefteq H$
 $\implies \text{graph } F f \subseteq \text{graph } H h \implies \forall x \in H. h x \leq p x$
 $\implies \text{graph } H h \in \text{norm-pres-extensions } E p F f$
<proof>

lemma *norm-pres-extensionI*:

$\exists H h. g = \text{graph } H h$
 $\wedge \text{linearform } H h$
 $\wedge H \trianglelefteq E$
 $\wedge F \trianglelefteq H$
 $\wedge \text{graph } F f \subseteq \text{graph } H h$
 $\wedge (\forall x \in H. h x \leq p x) \implies g \in \text{norm-pres-extensions } E p F f$
<proof>

end

8 The norm of a function

theory *FunctionNorm* **imports** *NormedSpace FunctionOrder* **begin**

8.1 Continuous linear forms

A linear form f on a normed vector space $(V, \|\cdot\|)$ is *continuous*, iff it is bounded, i.e.

$$\exists c \in \mathbb{R}. \forall x \in V. |f x| \leq c \cdot \|x\|$$

In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

locale *continuous* = *var* V + *norm-syntax* + *linearform* +
assumes *bounded*: $\exists c. \forall x \in V. |f x| \leq c * \|x\|$

declare *continuous.intro* [*intro?*] *continuous-axioms.intro* [*intro?*]

lemma *continuousI* [*intro*]:
includes *norm-syntax* + *linearform*
assumes $r: \bigwedge x. x \in V \implies |f x| \leq c * \|x\|$
shows *continuous* V *norm* f
<proof>

8.2 The norm of a linear form

The least real number c for which holds

$$\forall x \in V. |f x| \leq c \cdot \|x\|$$

is called the *norm* of f .

For non-trivial vector spaces $V \neq \{0\}$ the norm can be defined as

$$\|f\| = \sup_{x \neq 0} |f x| / \|x\|$$

For the case $V = \{0\}$ the supremum would be taken from an empty set. Since \mathbb{R} is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be $\{ \} \geq 0$ so that *fn-norm* has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be 0 , as all other elements are $\{ \} \geq 0$.

Thus we define the set B where the supremum is taken from as follows:

$$\{0\} \cup \{|f x| / \|x\|. \ x \neq 0 \wedge x \in V\}$$

fn-norm is equal to the supremum of B , if the supremum exists (otherwise it is undefined).

locale *fn-norm* = *norm-syntax* +
fixes B **defines** $B \ V f \equiv \{0\} \cup \{|f x| / \|x\| \mid x. x \neq 0 \wedge x \in V\}$
fixes *fn-norm* ($\|\cdot\|$) -- [$0, 1000$] 999
defines $\|f\|_V \equiv \bigsqcup (B \ V f)$

lemma (in *fn-norm*) *B-not-empty* [intro]: $0 \in B \ V \ f$
 ⟨proof⟩

The following lemma states that every continuous linear form on a normed space $(V, \|\cdot\|)$ has a function norm.

lemma (in *normed-vectorspace*) *fn-norm-works*:
 includes *fn-norm* + *continuous*
 shows $\text{lub } (B \ V \ f) (\|f\| - V)$
 ⟨proof⟩

lemma (in *normed-vectorspace*) *fn-norm-ub* [iff?]:
 includes *fn-norm* + *continuous*
 assumes $b: b \in B \ V \ f$
 shows $b \leq \|f\| - V$
 ⟨proof⟩

lemma (in *normed-vectorspace*) *fn-norm-leastB*:
 includes *fn-norm* + *continuous*
 assumes $b: \bigwedge b. b \in B \ V \ f \implies b \leq y$
 shows $\|f\| - V \leq y$
 ⟨proof⟩

The norm of a continuous function is always ≥ 0 .

lemma (in *normed-vectorspace*) *fn-norm-ge-zero* [iff]:
 includes *fn-norm* + *continuous*
 shows $0 \leq \|f\| - V$
 ⟨proof⟩

The fundamental property of function norms is:

$$|f \ x| \leq \|f\| \cdot \|x\|$$

lemma (in *normed-vectorspace*) *fn-norm-le-cong*:
 includes *fn-norm* + *continuous* + *linearform*
 assumes $x: x \in V$
 shows $|f \ x| \leq \|f\| - V * \|x\|$
 ⟨proof⟩

The function norm is the least positive real number for which the following inequation holds:

$$|f \ x| \leq c \cdot \|x\|$$

lemma (in *normed-vectorspace*) *fn-norm-least* [intro?]:
 includes *fn-norm* + *continuous*
 assumes *ineq*: $\forall x \in V. |f \ x| \leq c * \|x\|$ and *ge*: $0 \leq c$
 shows $\|f\| - V \leq c$
 ⟨proof⟩

end

9 Zorn's Lemma

theory *ZornLemma* **imports** *Zorn* **begin**

Zorn's Lemma states: if every linear ordered subset of an ordered set S has an upper bound in S , then there exists a maximal element in S . In our application, S is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn's lemma can be modified: if S is non-empty, it suffices to show that for every non-empty chain c in S the union of c also lies in S .

theorem *Zorn's-Lemma*:

assumes r : $\bigwedge c. c \in \text{chain } S \implies \exists x. x \in c \implies \bigcup c \in S$

and aS : $a \in S$

shows $\exists y \in S. \forall z \in S. y \subseteq z \longrightarrow y = z$

<proof>

end

Part II

Lemmas for the Proof

10 The supremum w.r.t. the function order

theory *HahnBanachSupLemmas* **imports** *FunctionNorm ZornLemma* **begin**

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let E be a real vector space with a seminorm p on E . F is a subspace of E and f a linear form on F . We consider a chain c of norm-preserving extensions of f , such that $\bigcup c = \text{graph } H \ h$. We will show some properties about the limit function h , i.e. the supremum of the chain c .

Let c be a chain of norm-preserving extensions of the function f and let $\text{graph } H \ h$ be the supremum of c . Every element in H is member of one of the elements of the chain.

lemmas $[\text{dest?}] = \text{chainD}$

lemmas $\text{chainE2} [\text{elim?}] = \text{chainD2} [\text{elim-format}, \text{standard}]$

lemma *some- $H'h'$* :

assumes $M: M = \text{norm-pres-extensions } E \ p \ F \ f$

and $cM: c \in \text{chain } M$

and $u: \text{graph } H \ h = \bigcup c$

and $x: x \in H$

shows $\exists H' \ h'. \text{graph } H' \ h' \in c$

$\wedge (x, h \ x) \in \text{graph } H' \ h'$

$\wedge \text{linearform } H' \ h' \wedge H' \trianglelefteq E$

$\wedge F \trianglelefteq H' \wedge \text{graph } F \ f \subseteq \text{graph } H' \ h'$

$\wedge (\forall x \in H'. \ h' \ x \leq p \ x)$

$\langle \text{proof} \rangle$

Let c be a chain of norm-preserving extensions of the function f and let $\text{graph } H \ h$ be the supremum of c . Every element in the domain H of the supremum function is member of the domain H' of some function h' , such that h extends h' .

lemma *some- $H'h'$* :

assumes $M: M = \text{norm-pres-extensions } E \ p \ F \ f$

and $cM: c \in \text{chain } M$

and $u: \text{graph } H \ h = \bigcup c$

and $x: x \in H$

shows $\exists H' \ h'. x \in H' \wedge \text{graph } H' \ h' \subseteq \text{graph } H \ h$

$\wedge \text{linearform } H' \ h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$

$\wedge \text{graph } F \ f \subseteq \text{graph } H' \ h' \wedge (\forall x \in H'. \ h' \ x \leq p \ x)$

$\langle \text{proof} \rangle$

Any two elements x and y in the domain H of the supremum function h are both in the domain H' of some function h' , such that h extends h' .

lemma *some- $H'h'2$* :

assumes $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$
and $cM: c \in \text{chain } M$
and $u: \text{graph } H \text{ } h = \bigcup c$
and $x: x \in H$
and $y: y \in H$
shows $\exists H' \text{ } h'. x \in H' \wedge y \in H'$
 $\wedge \text{graph } H' \text{ } h' \subseteq \text{graph } H \text{ } h$
 $\wedge \text{linearform } H' \text{ } h' \wedge H' \leq E \wedge F \leq H'$
 $\wedge \text{graph } F \text{ } f \subseteq \text{graph } H' \text{ } h' \wedge (\forall x \in H'. h' \text{ } x \leq p \text{ } x)$
 $\langle \text{proof} \rangle$

The relation induced by the graph of the supremum of a chain c is definite, i. e. t is the graph of a function.

lemma *sup-definite*:

assumes $M\text{-def}: M \equiv \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$
and $cM: c \in \text{chain } M$
and $xy: (x, y) \in \bigcup c$
and $xz: (x, z) \in \bigcup c$
shows $z = y$
 $\langle \text{proof} \rangle$

The limit function h is linear. Every element x in the domain of h is in the domain of a function h' in the chain of norm preserving extensions. Furthermore, h is an extension of h' so the function values of x are identical for h' and h . Finally, the function h' is linear by construction of M .

lemma *sup-lf*:

assumes $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$
and $cM: c \in \text{chain } M$
and $u: \text{graph } H \text{ } h = \bigcup c$
shows $\text{linearform } H \text{ } h$
 $\langle \text{proof} \rangle$

The limit of a non-empty chain of norm preserving extensions of f is an extension of f , since every element of the chain is an extension of f and the supremum is an extension for every element of the chain.

lemma *sup-ext*:

assumes $\text{graph}: \text{graph } H \text{ } h = \bigcup c$
and $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$
and $cM: c \in \text{chain } M$
and $ex: \exists x. x \in c$
shows $\text{graph } F \text{ } f \subseteq \text{graph } H \text{ } h$
 $\langle \text{proof} \rangle$

The domain H of the limit function is a superspace of F , since F is a subset of H . The existence of the 0 element in F and the closure properties follow from the fact that F is a vector space.

lemma *sup-supF*:

assumes $\text{graph}: \text{graph } H \text{ } h = \bigcup c$
and $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$
and $cM: c \in \text{chain } M$

```

and  $ex: \exists x. x \in c$ 
and  $FE: F \trianglelefteq E$ 
shows  $F \trianglelefteq H$ 
<proof>

```

The domain H of the limit function is a subspace of E .

```

lemma sup-subE:
assumes graph:  $\text{graph } H \ h = \bigcup c$ 
and  $M: M = \text{norm-pres-extensions } E \ p \ F \ f$ 
and  $cM: c \in \text{chain } M$ 
and  $ex: \exists x. x \in c$ 
and  $FE: F \trianglelefteq E$ 
and  $E: \text{vector space } E$ 
shows  $H \trianglelefteq E$ 
<proof>

```

The limit function is bounded by the norm p as well, since all elements in the chain are bounded by p .

```

lemma sup-norm-pres:
assumes graph:  $\text{graph } H \ h = \bigcup c$ 
and  $M: M = \text{norm-pres-extensions } E \ p \ F \ f$ 
and  $cM: c \in \text{chain } M$ 
shows  $\forall x \in H. h \ x \leq p \ x$ 
<proof>

```

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma *abs-HahnBanach* (see page 23). For real vector spaces the following inequations are equivalent:

$$\forall x \in H. |h \ x| \leq p \ x \quad \text{and} \quad \forall x \in H. h \ x \leq p \ x$$

```

lemma abs-ineq-iff:
includes subspace  $H \ E + \text{vector space } E + \text{seminorm } E \ p + \text{linearform } H \ h$ 
shows  $(\forall x \in H. |h \ x| \leq p \ x) = (\forall x \in H. h \ x \leq p \ x)$  (is ?L = ?R)
<proof>

```

end

11 Extending non-maximal functions

theory *HahnBanachExtLemmas* **imports** *FunctionNorm* **begin**

In this section the following context is presumed. Let E be a real vector space with a seminorm q on E . F is a subspace of E and f a linear function on F . We consider a subspace H of E that is a superspace of F and a linear form h on H . H is not equal to E and x_0 is an element in $E - H$. H is extended to the direct sum $H' = H + \text{lin } x_0$, so for any $x \in H'$ the decomposition of $x = y + a \cdot x$ with $y \in H$ is unique. h' is defined on H' by $h' \ x = h \ y + a \cdot \xi$ for a certain ξ .

Subsequently we show some properties of this extension h' of h .

This lemma will be used to show the existence of a linear extension of f (see page ??). It is a consequence of the completeness of \mathbb{R} . To show

$$\exists \xi. \forall y \in F. a y \leq \xi \wedge \xi \leq b y$$

it suffices to show that

$$\forall u \in F. \forall v \in F. a u \leq b v$$

lemma *ex-xi*:

includes *vectorspace* F
assumes $r: \bigwedge u v. u \in F \implies v \in F \implies a u \leq b v$
shows $\exists xi::real. \forall y \in F. a y \leq xi \wedge xi \leq b y$
 $\langle proof \rangle$

The function h' is defined as a $h' x = h y + a \cdot \xi$ where $x = y + a \cdot \xi$ is a linear extension of h to H' .

lemma *h'-lf*:

includes $var H + var h + var E$
assumes $h'\text{-def}: h' \equiv \lambda x. let (y, a) =$
 $SOME (y, a). x = y + a \cdot x0 \wedge y \in H in h y + a * xi$
and $H'\text{-def}: H' \equiv H + lin x0$
and $HE: H \leq E$
includes *linearform* $H h$
assumes $x0: x0 \notin H \ x0 \in E \ x0 \neq 0$
includes *vectorspace* E
shows *linearform* $H' h'$
 $\langle proof \rangle$

The linear extension h' of h is bounded by the seminorm p .

lemma *h'-norm-pres*:

includes $var H + var h + var E$
assumes $h'\text{-def}: h' \equiv \lambda x. let (y, a) =$
 $SOME (y, a). x = y + a \cdot x0 \wedge y \in H in h y + a * xi$
and $H'\text{-def}: H' \equiv H + lin x0$
and $x0: x0 \notin H \ x0 \in E \ x0 \neq 0$
includes *vectorspace* $E + subspace H E + seminorm E p + linearform H h$
assumes $a: \forall y \in H. h y \leq p y$
and $a': \forall y \in H. -p (y + x0) - h y \leq xi \wedge xi \leq p (y + x0) - h y$
shows $\forall x \in H'. h' x \leq p x$
 $\langle proof \rangle$

end

Part III

The Main Proof

12 The Hahn-Banach Theorem

theory *HahnBanach* **imports** *HahnBanachLemmas* **begin**

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

12.1 The Hahn-Banach Theorem for vector spaces

Hahn-Banach Theorem. Let F be a subspace of a real vector space E , let p be a semi-norm on E , and f be a linear form defined on F such that f is bounded by p , i.e. $\forall x \in F. f\ x \leq p\ x$. Then f can be extended to a linear form h on E such that h is norm-preserving, i.e. h is also bounded by p .

Proof Sketch.

1. Define M as the set of norm-preserving extensions of f to subspaces of E . The linear forms in M are ordered by domain extension.
2. We show that every non-empty chain in M has an upper bound in M .
3. With Zorn's Lemma we conclude that there is a maximal function g in M .
4. The domain H of g is the whole space E , as shown by classical contradiction:
 - Assuming g is not defined on whole E , it can still be extended in a norm-preserving way to a super-space H' of H .
 - Thus g can not be maximal. Contradiction!

theorem *HahnBanach*:

includes *vectorspace* E + *subspace* F E + *seminorm* E p + *linearform* F f

assumes fp : $\forall x \in F. f\ x \leq p\ x$

shows $\exists h. \text{linearform } E\ h \wedge (\forall x \in F. h\ x = f\ x) \wedge (\forall x \in E. h\ x \leq p\ x)$

— Let E be a vector space, F a subspace of E , p a seminorm on E ,

— and f a linear form on F such that f is bounded by p ,

— then f can be extended to a linear form h on E in a norm-preserving way.

<proof>

12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form f and a seminorm p the following inequations are equivalent:¹

$$\forall x \in H. |h\ x| \leq p\ x \quad \text{and} \quad \forall x \in H. h\ x \leq p\ x$$

¹This was shown in lemma *abs-ineq-iff* (see page 21).

theorem *abs-HahnBanach*:

includes *vectorspace* E + *subspace* F E + *linearform* F f + *seminorm* E p

assumes fp : $\forall x \in F. |f\ x| \leq p\ x$

shows $\exists g. \text{linearform } E\ g$

$\wedge (\forall x \in F. g\ x = f\ x)$

$\wedge (\forall x \in E. |g\ x| \leq p\ x)$

$\langle \text{proof} \rangle$

12.3 The Hahn-Banach Theorem for normed spaces

Every continuous linear form f on a subspace F of a norm space E , can be extended to a continuous linear form g on E such that $\|f\| = \|g\|$.

theorem *norm-HahnBanach*:

includes *normed-vectorspace* E + *subspace* F E + *linearform* F f + *fn-norm* + *continuous* F *norm* $(\|-|\|)$ f

shows $\exists g. \text{linearform } E\ g$

$\wedge \text{continuous } E\ \text{norm } g$

$\wedge (\forall x \in F. g\ x = f\ x)$

$\wedge \|g\|_E = \|f\|_F$

$\langle \text{proof} \rangle$

end

References

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- [3] B. Nowak and A. Trybulec. Hahn-Banach theorem. *Journal of Formalized Mathematics*, 5, 1993. <http://mizar.uwb.edu.pl/JFM/Vol5/hahnban.html>.